On the recognition of $\frac{1}{2}$-hyperbolic graphs: a subcubic equivalence with the C_4-free graph recognition problem

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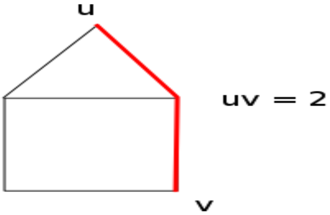
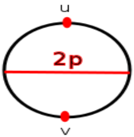
- graphs in this study are:
  - undirected;
  - simple;
  - **unweighted**;
  - and connected.

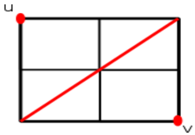


distance = length of a shortest-path
(an edge = a path of length 1)

A tree-like parameter

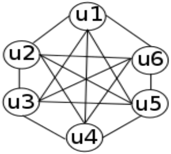
- which (maximum) gap between two u, v-shortest paths ?

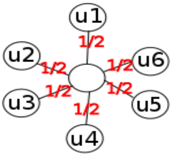




- Tree: Unique shortest-path $\longrightarrow$ hyperbolicity 0
- Conversely:

  0-hyperbolic graph $\longrightarrow$ (weighted) tree metric





Formal Definition

Definition (4-points Condition, [Gromov, 1987])

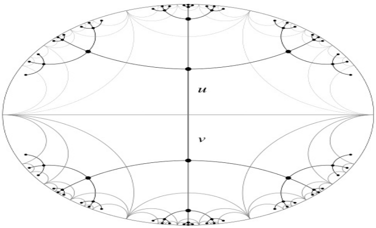
Say that graph G is δ-hyperbolic if, for every 4-tuple a, b, c, d of V, the two largest sums amongst
$S_1 = d(a,b) + d(c,d)$,
$S_2 = d(a,c) + d(b,d)$, and
$S_3 = d(a,d) + d(b,c)$ differ by at most 2δ.
The graph hyperbolicity, denoted $\delta(G)$, is the least δ such that G is δ-hyperbolic.

- Other "equivalent" definitions (δ-slim triangles, Gromov product, ...).

Some applications

- network security
- bioinformatics
- **routing**

→ **[Kleinberg, 2007]** embedding into a hyperbolic space
The resulting metric has to stay close to the metric of the graph.



Problem

Can we compute the hyperbolicity value efficiently ?

State of the art:

- naive enumeration of 4-tuples ($\Theta(n^4)$);

[Fournier et al., 2012]

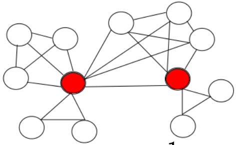
- reduction to (min, max) matrix product ($O(n^{3.69})$);

[Coudert et al., 2012]

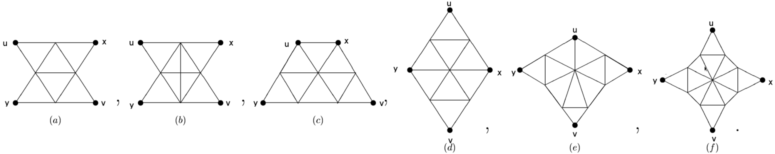
- sorting of 2-tuples + breaking rule ($O(n^4)$).

Graphs with small hyperbolicity value

- Motivation: complex networks often have a bounded hyperbolicity value .
- **[Bandelt et Mulder, 1986]** 0-hyperbolic graphs: biconnected components = complete subgraphs
  → linear-time recognition



- **[Bandelt et Chepoi, 2003]** $\frac{1}{2}$-hyperbolic graphs: *Forbidden* (isometric) subgraphs
  - no cycle with length $\neq 3, 5$ + six other subgraphs.



(a), (b), (c), (d), (e), (f).

Our main contribution

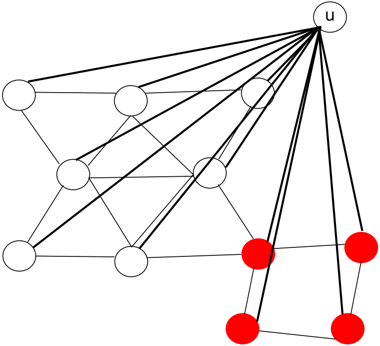
The complexity of deciding if a graph is $\frac{1}{2}$-hyperbolic is equivalent to the complexity of finding an (induced) C_4 in a graph.

- both problems are reducible to fast rectangular matrix multiplication ($O(n^{\omega(1,1,\log_n m)}) = O(n^{3.333953})$.

C_4-free detection $\propto \frac{1}{2}$-hyperbolic recognition

- addition of a universel vertex

→ linear-time reduction



(using **[Bandelt et Chepoi, 2003]**) C_4 is the only forbidden subgraph in such case.

$\frac{1}{2}$-hyperbolic recognition $\propto$ C_4-free detection (1/3)

- Quickly excluding large (isometric) cycles:

using a constant-factor approximation for hyperbolicity

- $O(n^{2.69})$ **[Fournier et al., 2012]**;
- $\tilde{O}(n^{2.37})$ **[Chalopin et al., 2013]**.

Given a c-factor approximation, denoted $\delta_c(G)$:

- If $\delta_c(G) > \frac{c}{2}$, then G is not $\frac{1}{2}$-hyperbolic;
- Otherwise, G does not contain isometric cycles with length $\geq \Theta(c)$.

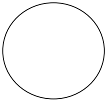
$\frac{1}{2}$-hyperbolic recognition $\propto$ C_4-free detection (2/3)

- Excluding the isometric cycles with length bounded by $O(c)$:

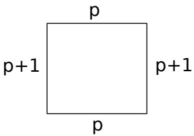
using graph *powers* $G = G^1, G^2, G^3, \ldots, G^{O(c)}$.

an edge in G^i = a path of length (at most) i in G.

$\rightarrow$ computable in time $\tilde{O}(n^{2.37})$



isometric C_4p+2 in G

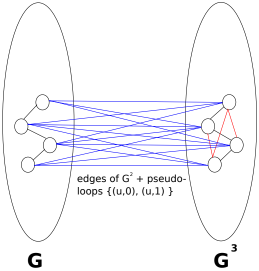


induced C_4 in G^2p

- a simple computation shows that every G^i has to be C_4-free so that G is $\frac{1}{2}$-hyperbolic.

$\frac{1}{2}$-hyperbolic recognition $\propto$ C_4-free detection (3/3)

- Excluding the forbidden (isometric) subgraphs: using *modified* graph powers $G^{[2]}, \ldots, G^{[O(c)]}$.



We use the characterization of **[Bandelt et Chepoi, 2003]** for the correctness.

Final Result

- There is a linear-time reduction from the C_4-free graph recognition problem to the $\frac{1}{2}$-hyperbolic graph recognition problem.

- There is a subcubic-time reduction from the $\frac{1}{2}$-hyperbolic graph recognition problem to the C_4-free graph recognition problem.

Finding an induced C_4

- Basic idea: G is C_4-free if, and only if, for every pair $u, v \in V$ at a distance $uv = 2$ we have $N(u) \cap N(v)$ that is a clique.

This can be tested using two matrix multiplications:

- If A is the adjacency matrix of G, then $A^2_{u,v} = |N(u) \cap N(v)|$;
- If T is s.t. $T_{u,e} = \delta_{e \in N(u)}$, then $TT^\top_{u,v} = |\{e \in E : e \subseteq N(u) \cap N(v)|$;
- One has to check for every pair $u, v \in V$ at a distance 2 that we have $TT^\top_{u,v} = \frac{A^2_{u,v}(A^2_{u,v}-1)}{2}$.

The time complexity is $O(n^{\omega(1,1,\log_n m)})$ (using **[Huang et al., 1998]**).

Conclusion and Perspectives

- Our results:
  - A subcubic equivalence between a *metric* problem (the recognition of a $\frac{1}{2}$-hyperbolic graph) and a *structural* problem (*e.g.* the detection of a C_4)
  - An improved algorithm in time $O(n^{\omega(1,1,\log_n m)}) = O(n^{3.333953}) = o(n^{3.69})$
- Future work: extending our results to 1-hyperbolic graphs

$\longrightarrow$ this would yield a 4-factor approximation for the hyperbolicity

Sources (1/2)

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- **[Coudert et al., 2012]**
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Sources (2/2)

- **[Fournier et al., 2012]**
  Fournier, H., Ismail, A., Vigneron, A. (2012). Computing the Gromov hyperbolicity of a discrete metric space. *arXiv preprint* arXiv:1210.3323.
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Questions